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Oscillations of Difference Equations with Several Delays

B. G. ZHANG
Department of Applied Mathematics, Ocean University of Qingdao
Qingdao, 266003, P.R. China

YONG ZHOU
Department of Mathematics, Xiangtan University
Hunan 411105, P.R. China

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Abstract—Consider the first-order delay difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} = 0, \qquad n = 0, 1, 2, \dots,$$

where $\liminf_{n\to\infty} P_i(n) = p_i \geq 0$, $k_i > 0$, $i = 1, 2, \dots$. Sufficient conditions for the oscillation of all solutions of the above equation are established in the case when the corresponding "limiting" equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0, \qquad n = 0, 1, 2, \dots,$$

admits nonoscillatory solutions. Oscillation criteria for the nonlinear difference equation are also derived as applications.

Keywords—Oscillation, Difference equations, Delay.

1. INTRODUCTION

We consider the first-order difference equation with several delays

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} = 0, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where $\{P_i(n)\}_{n=1}^{\infty}$ is a real sequence with $P_i(n) \geq 0$ for all large n, $k_i > 0$ $(i = 1, 2, \dots, m)$ are integers, and

$$\liminf_{n\to\infty} P_i(n) = p_i \geq 0, \qquad i = 1, 2, \dots, m. \tag{2}$$

Then the corresponding limiting equation is

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0, \qquad n = 0, 1, 2, \dots, \tag{3}$$

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with characteristic equation

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0. \tag{4}$$

It is well known (for example, see [1]) that all solutions of equation (3) oscillate if and only if (4) has no positive roots. In [1], Györi and Ladas showed that all solutions of equation (1) oscillate if all solutions of equation (3) oscillate. However, the following situation is also possible: *all solutions of equation (1) oscillate in spite of the fact that the corresponding limiting equation (3) admits nonoscillatory solutions.*

Oscillatory properties of equations with one delay only of the form

$$x_{n+1} - x_n + P(n) x_{n-k} = 0, \qquad n = 0, 1, 2, \dots, \tag{5}$$

where $P(n)$ is a sequence with $P(n) \geq 0$, $k \geq 0$ is a integer, in the case

$$\liminf_{n \to \infty} P(n) = p \leq \frac{k^k}{(k+1)^{k+1}}, \tag{6}$$

have been investigated by Li [2], Stavroulakis [3], Zhang and Tian [4], and Zhou and Wang [5]. Other oscillation results for (1) and (5) can be found in [1,6–12] and the references cited therein.

In this paper, we introduce some new techniques to establish sufficient conditions for the oscillation of all solutions of equation (1) in the case when the corresponding limiting equation (3) admits nonoscillatory solutions. It is to be pointed out that there is no result in this case for a delay difference equation with several delays. As applications, we also obtain oscillation criteria for nonlinear difference equations with several delays.

By a solution of equation (1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k^*$ where $k^* = \max_{i \geq 1}\{k_i\}$, and which satisfies (1) for $n \geq 0$. A solution $\{x_n\}$ of (1) is said to be oscillatory if the terms x_n of the sequence $\{x_n\}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

2. MAIN RESULTS

First, we define a sequence $\{\lambda_l\}_{l=1}^{\infty}$ by

$$\lambda_1 = 1, \qquad \lambda_{l+1} = 1 - \sum_{i=1}^{m} p_i \lambda_l^{-k_i}, \qquad l = 1, 2, \dots, \tag{7}$$

where $p_i \geq 0$, $i = 1, 2, \dots, m$.

The following lemma will be used to prove our main results.

LEMMA 1. *Assume that the sequence $\{\lambda_l\}$ is defined by equation (7). Then $\lambda_* \leq \lambda_l \leq 1$ and $\lim_{n \to \infty} \lambda_l = \lambda_*$, where λ_* is the largest root of equation (4) on $(0, 1]$.*

PROOF. Since $\lambda_1 = 1 \geq \lambda_*$, hence, by induction, we have

$$1 \geq \lambda_{l+1} = 1 - \sum_{i=1}^{m} p_i \lambda_l^{-k_i} \geq 1 - \sum_{i=1}^{m} p_i \lambda_*^{-k_i} = \lambda_*, \qquad l = 1, 2, \dots.$$

On the other hand, we find $\lambda_2 < 1 = \lambda_1$. By induction, we obtain

$$\lambda_{l+1} = 1 - \sum_{i=1}^{m} p_i \lambda_l^{-k_i} \leq 1 - \sum_{i=1}^{m} p_i \lambda_{l-1}^{-k_i} = \lambda_l, \qquad l = 1, 2, \dots.$$

Hence, $\{\lambda_l\}$ is nonincreasing and bounded. Therefore, $\lim_{l \to \infty} \lambda_l$ exists.

Letting $l \to \infty$ on (7), we have $\lim_{l \to \infty} \lambda_l = \lambda_*$. The proof is completed.

In the following, we consider linear difference inequalities of the form

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} \le 0, \qquad n = 0, 1, 2, \dots, \tag{8}$$

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} \ge 0, \qquad n = 0, 1, 2, \dots. \tag{9}$$

THEOREM 1. *Assume that (2) holds and (4) has positive roots. Further, assume that*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} P_i(n) \lambda_*^{-k_i} > \frac{1}{1+\lambda_*}, \tag{10}$$

where λ_* is the largest root of (4) on $(0, 1]$. Then

(i) *inequality (8) has no eventually positive solutions;*
(ii) *inequality (9) has no eventually negative solutions; and*
(iii) *every solution of equation (1) oscillates.*

PROOF. It is sufficient to prove (i)–(iii) follow from (i). Assume, for the sake of contradiction, that $\{x_n\}$ is an eventually positive solution of (1). Then, there exists $n_1 \ge 0$ such that $x_n > 0$ and $x_{n-k_i} > 0$, $i = 1, 2, \dots, m$, for $n \ge n_1$. Therefore, from (1), we have

$$x_n \ge x_{n+1}, \qquad \text{for } n \ge n_1,$$

which gives

$$x_{n-k_i} \ge \lambda_1^{-k_i} x_n, \qquad \text{for } n \ge n_1 + k_i. \tag{11}$$

Using (11) and (1), we have

$$x_n \ge x_{n+1} + \sum_{i=1}^{m} p_i \lambda_1^{-k_i} x_n,$$

i.e.,

$$x_{n+1} \le \left(1 - \sum_{i=1}^{m} p_i \lambda_1^{-k_i}\right) x_n = \lambda_2 x_n, \qquad \text{for } n \ge n_1 + k_i,$$

which gives

$$x_{n-k_i} \ge \lambda_2^{-k_i} x_n, \qquad \text{for } n \ge n_1 + 2k_i.$$

Repeating the above procedure, we get

$$x_{n+1} \le \left(1 - \sum_{i=1}^{m} p_i \lambda_{l-1}^{-k_i}\right) x_n = \lambda_l x_n, \qquad \text{for } n \ge n_1 + (l-1)k_i. \tag{12}$$

Since $\lim_{n\to\infty} \lambda_l = \lambda_*$, for a sequence $\{\varepsilon_l\}$ with $\varepsilon_l > 0$ and $\varepsilon_l \to 0$ as $l \to \infty$, by (12), there exists a sequence $\{n_l\}$ such that $n_l \to \infty$ as $l \to \infty$ and

$$x_{n+1} \le (\lambda_* + \varepsilon_l) x_n, \qquad \text{for } n \ge n_l, \tag{13}$$

and

$$x_{n-k_i} \ge (\lambda_* + \varepsilon_l)^{-k_i} x_n, \qquad \text{for } n \ge n_l + k_i. \tag{14}$$

On the other hand, from (1) and (5), we have

$$x_n \ge \sum_{i=1}^{m} P_i(n) (\lambda_* + \varepsilon_l)^{-k_i+1} x_{n-1}, \qquad \text{for } n \ge n_l + k_i - 1, \tag{15}$$

which implies

$$\frac{x_{n+1}}{x_n} \geq \sum_{i=1}^{m} P_i(n)(\lambda_* + \varepsilon_l)^{-k_i+1}, \qquad \text{for } n \geq n_l + k_i. \tag{16}$$

From (1), (14), and (16), we obtain

$$\begin{aligned} 1 &\geq \frac{x_{n+1}}{x_n} + \sum_{i=1}^{m} P_i(n) \frac{x_{n-k_i}}{x_n} \\ &\geq \sum_{i=1}^{m} P_i(n)(\lambda_* + \varepsilon_l)^{-k_i+1} + \sum_{i=1}^{m} P_i(n)(\lambda_* + \varepsilon_l)^{-k_i}. \end{aligned} \tag{17}$$

Letting $l \to \infty$, (17) yields

$$1 \geq (1 + \lambda_*) \limsup_{n\to\infty} \sum_{i=1}^{m} P_i(n)\lambda_*^{-k_i}. \tag{18}$$

That is,

$$\limsup_{n\to\infty} \sum_{i=1}^{m} P_i(n)\lambda_*^{-k_i} \leq \frac{1}{1+\lambda_*},$$

which contradicts (10) and completes the proof.

It is easy to see that the characteristic equation

$$\lambda - 1 + p\lambda^{-k} = 0 \tag{19}$$

has positive roots if and only if (6) holds. By using Theorem 1, we obtain the following corollary.

COROLLARY 1. *Assume that (6) holds and*

$$\limsup_{n\to\infty} P(n) > \frac{\lambda_*}{1+\lambda_*}, \tag{20}$$

where λ_* *is the unique positive root of (19) on* $([(k+1)p]^{1/k}, 1]$. *Then every solution of equation (5) oscillates.*

EXAMPLE 1. Consider the delay difference equation

$$y_{n+1} - y_n + P(n)y_{n-1} = 0, \qquad n = 0, 1, 2, \dots. \tag{21}$$

Let $P(0) = 11/25$, $P(1) = 3/16$, $P(n+2) = P(n)$ for $n = 0, 1, 2, \dots$. Then

$$\liminf_{n\to\infty} P(n) = p = \frac{3}{16}$$

and

$$\limsup_{n\to\infty} P(n) = \frac{11}{25} > \frac{\lambda_*}{1+\lambda_*} = \frac{\left(1+\sqrt{1-4p}\right)/2}{1+\left(1+\sqrt{1-4p}\right)/2} = 0.4285\dots.$$

Thus, according to Corollary 1, every solution of (21) oscillates. But none of the results in [1–12] can be applied to this equation.

In the following, we consider the nonlinear difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) f_i(x_{n-k_i}) = 0, \qquad n = 0, 1, 2, \dots, \tag{22}$$

where

$$f \in C(R, R) \qquad \text{and} \qquad u f_i(u) > 0, \qquad \text{for } u \neq 0. \tag{23}$$

THEOREM 2. *Assume that (2) and (23) hold and that*

$$\liminf_{u \to 0} \frac{f_i(u)}{u} \geq 1, \qquad \text{for } i = 1, 2, \dots, m. \tag{24}$$

Further, assume that (4) has positive roots and (10) holds. Then every solution of equation (22) oscillates.

PROOF. Assume, for the sake of contradiction, that equation (22) has a nonoscillatory solution $\{x_n\}$. We assume that $\{x_n\}$ is eventually positive. The case where $\{x_n\}$ is eventually negative is similar and is omitted. It is not difficult to see that

$$\lim_{n \to \infty} x_n = 0. \tag{25}$$

By (24) and (25), we get

$$\liminf_{n \to \infty} \frac{f_i(x_{n-k_i})}{x_{n-k_i}} \geq 1, \qquad \text{for } i = 1, 2, \dots, m. \tag{26}$$

From (22) and (26), we have

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} \leq 0. \tag{27}$$

But by Theorem 1, when (2) and (10) hold, (27) cannot have eventually positive solutions. This contradiction completes the proof.

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